

Specimen examination paper

Candidates will be assessed on their 3 best answers from Section A and their 2 best answers from Section B.

Section A

1. Show that the function

$$w = \int_0^z \frac{d\zeta}{\sqrt{\zeta(1-\zeta^2)}}$$

maps the upper half-plane $\{\text{Im } z > 0\}$ onto a rectangle in the w -plane.

Using the fact that if $f(z)$ is analytic in a domain D containing a curve

C , then the length of the image of C is $\int_C |f'(z)| |dz|$ show that the

rectangle above is a square of side

$$\frac{1}{2} \int_0^1 t^{-3/4} (1-t)^{-1/2} dt.$$

2. From Green's lemma show that if $\theta \mapsto (u(\theta), v(\theta)) (0 \leq \theta \leq 2\pi)$ is a simple closed curve in the positive direction, then it encloses a domain whose area is

$$A = \frac{1}{2} \int_0^{2\pi} \left(u(\theta) \frac{dv(\theta)}{d\theta} - v(\theta) \frac{du(\theta)}{d\theta} \right) d\theta.$$

Hence, or otherwise, show that

$$\theta \mapsto \left(\cos\theta + \frac{1}{2} \cos 2\theta, \sin\theta + \frac{1}{2} \sin 2\theta \right) \quad (0 \leq \theta \leq 2\pi)$$

is a simple closed curve enclosing a domain of area $\frac{3\pi}{2}$.

3. Consider the differential equation

$$a_n w^{(n)}(z) + \dots + a_1 w'(z) + a_0 = 0$$

where the a_k are constants and $a_n \neq 0$. If $a_n \zeta^n + \dots + a_1 \zeta + a_0 = 0$ has distinct roots $\zeta_1, \dots, \zeta_n$ to show that given any solution $w(z)$ of the differential equation that is analytic in a neighbourhood of $z = 0$, there are constants $c_1, \dots, c_n$ such that

$$w(z) = \sum_{k=1}^n C_k e^{\zeta_k z}.$$

Find the general solution of

$$w'''(z) - w''(z) + w'(z) - 1 = 0.$$

4. Solve, using the Laplace transform,

$$y'' + w_0^2 y = \sin w t \quad (t > 0),$$

where w, w_0 are real constants with $w \neq w_0$ and $y(0) = y'(0) = 0$ and

$y(t), y'(t)$ behave suitably as $t \rightarrow \infty$.

Deduce the solution when $w = w_0 \neq 0$.

5. Suppose that $f(t)$ is bounded for $0 \leq t < \infty$ and

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad (|z| < 1).$$

Show that as $x \rightarrow \infty$ through real values,

$$\int_0^{\infty} e^{-xt} f(t) dt \sim \sum_{k=0}^{\infty} \frac{k! a_k}{x^{k+1}}.$$

6. If $P_n(x)$ is the n th Legendre polynomial show that

$$\int_{-1}^1 \log(1-x) \cdot P_n(x) dx = -\frac{2}{n(n+1)}.$$

Section B

7. Consider the equation

$$\nabla^2 u = \phi$$

for u in the unit disc, where ϕ is harmonic and real-valued. By considering the form of ϕ and the form of the Laplace operator in polars show that there is a function Φ harmonic in the disc such that $u - r^2 \Phi$ satisfies Laplace's equation. Hence show that any solution u is of the Goursat form,

$$u(z) = \operatorname{Re}[\bar{z} f(z) + g(z)],$$

where $f(z), g(z)$ are analytic in the disc.

8. Show that the general solution of the differential equation

$$\frac{d^2 y}{dx^2} + \frac{1}{3} xy = 0$$

is

$$y = \sqrt{x} \{A J_{1/3}(\xi) + B J_{-1/3}(\xi)\},$$

where $\xi^2 = \frac{4x^3}{27}$ and A, B are arbitrary constants.

9. Using the Laplace transform solve the Abel integral equation

$$t = \int_0^t \frac{y(u)}{(t-u)^\alpha} du \quad (0 < \alpha < 1)$$

for y .

10. Use the method of steepest descent to show that if C is any simple closed contour round the origin in the positive direction and n is a positive integer, then

$$I_n(x) = \frac{e^{-in\frac{\pi}{2}}}{2\pi i} \int_C \zeta^{-n-1} e^{\frac{i x}{2}(\zeta - \frac{1}{\zeta})} d\zeta$$

$$\sim \frac{e^x}{\sqrt{2\pi x}}$$

as $x \rightarrow \infty$ through real values.

Specimen examination paper

Solutions

References to Dettman's book will be indicated by JWD followed by page or section number.

1. From JWD, §6.5 and the discussion on pp.264-265 one sees that $z \mapsto w(z)$ maps $\{\operatorname{Im} z > 0\}$ onto a quadrilateral each of whose angles is $\frac{\pi}{2}$, i.e. onto a rectangle.

The segments $[-1,0], [0,1]$ in the z -plane map onto adjacent sides of the rectangle and these sides are of lengths

$$\int_{-1}^0 \frac{dt}{[|t|(1-t^2)]^{1/2}}, \quad \int_0^1 \frac{dt}{[t(1-t^2)]^{1/2}}$$

respectively, since

$$\left| \frac{dw}{dz} \right| = \frac{1}{|z(1-z^2)|^{1/2}}.$$

These two integrals have the same value and so the rectangle must be a square.

Putting $t^2 = u$ one sees that the length of side is

$$\int_0^1 \frac{dt}{[t(1-t)^2]^{1/2}} = \int_0^1 \frac{1}{u^{1/4}(1-u)^{1/2}} \frac{1}{2} \frac{du}{u^{1/2}}$$

$$= \frac{1}{2} \int_0^1 \frac{du}{u^{3/4}(1-u)^{1/2}}$$

2. In JWD, Theorem 3.1.2 take $P(x,y) = -y$, $Q(x,y) = x$ and then

$$\int_C -y dx + x dy = 2 \iint_R dx dy = 2A,$$

where A is the area of R . If C is parametrised by

$t \mapsto (x(t), y(t)) (a \leq t \leq b)$, then

$$\int_C -y dx + x dy = \int_a^b [-y(t) \frac{dx(t)}{dt} + x(t) \frac{dy(t)}{dt}] dt$$

and the result in the first part of the question follows in a different notation.

For the curve of the second part of the question note that

$$u(\theta) + i v(\theta) = e^{i\theta} + \frac{1}{2} e^{2i\theta}$$

and

$$(e^{i\theta_1} + \frac{1}{2} e^{2i\theta_1}) - (e^{i\theta_2} + \frac{1}{2} e^{2i\theta_2})$$

$$= (e^{i\theta_2} - e^{i\theta_1}) [1 + \frac{1}{2}(e^{i\theta_1} + e^{i\theta_2})]$$

For $0 \leq \theta_1, \theta_2 < 2\pi$, $e^{i\theta_2} = e^{i\theta_1} \Rightarrow \theta_2 = \theta_1$ and

$$1 + \frac{1}{2}(e^{i\theta_1} + e^{i\theta_2}) = 0 \Rightarrow e^{i\theta_2} = e^{i\theta_1} = -1 \Rightarrow \theta_2 = \theta_1 = \pi.$$

Therefore the curve is simple.

Now

$$u(\theta) = \cos\theta + \frac{1}{2} \cos 2\theta, \quad u'(\theta) = -\sin\theta - \sin 2\theta,$$

$$v(\theta) = \sin\theta + \frac{1}{2} \sin 2\theta, \quad v'(\theta) = \cos\theta + \cos 2\theta.$$

and so

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} [\cos\theta + \frac{1}{2} \cos 2\theta](\cos\theta + \cos 2\theta) + (\sin\theta + \frac{1}{2} \sin 2\theta)(\sin\theta + \sin 2\theta) d\theta \\ &= \frac{1}{2} (\pi + \frac{\pi}{2} + \pi + \frac{\pi}{2}) \\ &= \frac{3\pi}{2}, \end{aligned}$$

where we have used the orthonormal properties of sine and cosine as given in JWD, p.351.

3. If $w(z) = e^{\zeta_0 z}$, then $w^{(k)}(z) = \zeta_0^k e^{\zeta_0 z}$ and so

$$a_n w^{(n)} + \dots + a_0 = (a_n \zeta_0^n + \dots + a_0) e^{\zeta_0 z} = 0$$

provided ζ_0 is a root of $a_n \zeta^n + \dots + a_0 = 0$. From the hypotheses of

the question it follows that $e^{\zeta_k z}$ ($1 \leq k \leq n$) are solutions of the equation.

Let $w(z)$ be a solution that is analytic at $z = 0$ and consider the equations.

$$w(0) = C_1 + \dots + C_n$$

$$w'(0) = C_1 \zeta_1 + \dots + C_n \zeta_n$$

⋮

$$w(0) = C_1 \zeta_1 + \dots + C_n \zeta_n$$

for the 'unknowns' $C_1, \dots, C_n$. The determinant of the system is

$$\begin{vmatrix} 1 & \dots & 1 \\ \zeta_1 & \zeta_2 & \zeta_n \\ \dots & \dots & \dots \\ \zeta_1^{n+1} & \zeta_2^{n+1} & \zeta_n^{n+1} \end{vmatrix} = \pm \prod_{1 \leq j < k \leq n} (\zeta_j - \zeta_k) \neq 0,$$

since the ζ_k are distinct. Hence we can solve for $C_1, \dots, C_n$ and with this solution consider

$$W(z) = w(z) - \sum_{k=1}^n C_k e^{\zeta_k z}.$$

Now $W(z)$ is a solution of the equation and $W^{(k)}(0) = 0$ for

$0 \leq k \leq n-1$ and so $W^{(n)}(0) = 0$. Also, differentiating,

$$a_n W^{(n+1)}(z) + \dots + a_1 W^{(2)}(z) = 0$$

and so $W^{(n+1)}(0) = 0$; and hence $W^{(k)}(0) = 0$ ($k \geq 0$). Therefore

$$W(z) = \sum_{k=0}^{\infty} \frac{W^{(k)}(0)}{k!} z^k \equiv 0$$

and

$$w(z) = \sum_{k=1}^n C_k e^{\zeta_k z}.$$

For the second part of the question note that

$$\begin{aligned} s^3 - s^2 + s - 1 &= (s-1)(s^2+1) \\ &= (s-1)(s-i)(s+i) \end{aligned}$$

and so the general solution in this case is

$$C_1 e^z + C_2 e^{iz} + C_3 e^{-iz}$$

or

$$A e^z + B \cos z + C \sin z.$$

4. Taking the Laplace transform of the equation we have

$$\int_0^{\infty} y''(t) e^{-st} dt + w_0^2 Y(s) = \int_0^{\infty} \sin w t \cdot e^{-st} dt$$

where $Y(s) = \int_0^{\infty} y(t) e^{-st} dt$ is the Laplace transform of $y(t)$.

Also

$$\begin{aligned} \int_0^{\infty} y''(t) e^{-st} dt &= y'(t) \cdot e^{-st} \Big|_0^{\infty} + s \int_0^{\infty} y'(t) e^{-st} dt \\ &= s \cdot y(t) e^{-st} \Big|_0^{\infty} + s^2 \cdot \int_0^{\infty} y(t) e^{-st} dt \\ &= s^2 Y(s); \end{aligned}$$

and, by JWD, Example 9.1.1,

$$\int_0^{\infty} \sin wt e^{-st} dt = \frac{w}{s^2 + w^2}.$$

Hence

$$\begin{aligned} Y(s) &= \frac{w}{(s^2 + w_0^2)(s^2 + w^2)} \\ &= \frac{w}{w^2 - w_0^2} \left[\frac{1}{s^2 - w_0^2} - \frac{1}{s^2 + w^2} \right]. \end{aligned}$$

Inverting and noting Example 9.1.1 we obtain

$$y(t) = \frac{w}{w^2 - w_0^2} \left[\frac{1}{w_0} \sin w_0 t - \frac{1}{w} \sin w t \right].$$

Consider, finally,

$$y(t) = \frac{w}{w+w_0} \cdot - \frac{\frac{1}{w} \sin w t - \frac{1}{w_0} \sin w_0 t}{w-w_0}$$

and let $w \rightarrow w_0$ to obtain

$$\begin{aligned} y(t) &= - \frac{1}{2} \frac{d}{dw} \left(\frac{1}{w} \sin w t \right) \Big|_{w=w_0} \\ &= - \frac{1}{2} \left(\frac{t \cos w t}{w} - \frac{\sin w t}{w^2} \right) \Big|_{w=w_0} \\ &= \frac{1}{2} \left(\frac{\sin w_0 t}{w_0^2} - \frac{t \cos w_0 t}{w_0} \right) \end{aligned}$$

as the solution when $w = w_0 \neq 0$.

5. Consider

$$\int_0^\infty e^{-xt} f(t) dt = \left(\int_0^\epsilon + \int_\epsilon^\infty \right) e^{-xt} f(t) dt,$$

where $\epsilon > 0$ is 'small'. If $|f(t)| \leq M$ ($t \geq 0$) then, for $x > 0$,

$$\begin{aligned} \left| \int_\epsilon^\infty e^{-xt} f(t) dt \right| &\leq M \cdot \int_\epsilon^\infty e^{-xt} dt \\ &= \frac{M e^{-\epsilon x}}{x}. \end{aligned}$$

Also

$$\begin{aligned} \int_0^\epsilon e^{-xt} f(t) dt &= \sum_{k=0}^N a_k \int_0^\epsilon e^{-xt} t^k dt \\ &\quad + \int_0^\epsilon e^{-xt} \left(f(t) - \sum_{k=0}^N a_k t^k \right) dt. \end{aligned}$$

Assume that the radius of convergence of the series for $f(t)$ exceeds 2ε

and then, by the properties of power series,

$$|f(t) - \sum_{k=0}^N a_k t^k| \leq K(\varepsilon) \cdot t^{N+1} \quad (0 \leq t \leq \varepsilon).$$

Consequently, for $x > 0$,

$$\begin{aligned} \left| \int_0^\varepsilon e^{-xt} (f(t) - \sum_{k=0}^N a_k t^k) dt \right| &\leq K(\varepsilon) \cdot \int_0^\varepsilon e^{-xt} t^{N+1} dt \\ &= \frac{K(\varepsilon)}{x^{N+2}} \int_0^\infty e^{-y} u^{N+1} du \\ &= \frac{K(\varepsilon) \cdot (N+1)!}{x^{N+2}}. \end{aligned}$$

We have as well, for $x > 0$,

$$\int_0^\varepsilon e^{-xt} t^k dt = \int_0^\infty e^{-xt} t^k dt - \int_\varepsilon^\infty e^{-xt} t^k dt$$

and

$$\int_0^\infty e^{-xt} t^k dt = \frac{k!}{x^{k+1}};$$

$$\begin{aligned} \int_\varepsilon^\infty e^{-xt} t^k dt &\leq e^{-\frac{\varepsilon x}{2}} \int_0^\infty e^{-\frac{xt}{2}} t^k dt \\ &= e^{-\frac{\varepsilon x}{2}} \cdot k! \left(\frac{2}{x}\right)^{k+1}. \end{aligned}$$

From the above we see that, for $x > 0$,

$$\int_0^\infty e^{-xt} f(t) dt = \sum_{k=0}^N \frac{k! a_k}{x^{k+1}} + R_N,$$

where

$$|R_N| \leq \frac{Me^{-\epsilon x}}{x} + \frac{K(\epsilon) \cdot (N+1)!}{x^{N+2}} + \left(\sum_{k=0}^N k! \left(\frac{2}{x}\right)^{k+1} |a_k| \right) e^{-\frac{\epsilon x}{2}}$$

$$= O\left(\frac{1}{x^{N+2}}\right) \quad (x \rightarrow \infty).$$

Therefore result follows.

6. From Rodrigues' formula, JWD, p.319,

$$\int_{-1}^1 \log(1-x) \cdot P_n(x) dx = \frac{1}{2^n n!} \int_{-1}^1 \log(1-x) \cdot \frac{d^n}{dx^n} (x^2-1)^n \cdot dx.$$

Integrating by parts

$$\begin{aligned} \int_{-1}^1 \log(1-x) \cdot \frac{d^n}{dx^n} (x^2-1)^n \cdot dx &= \log(1-x) \cdot \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \Big|_{-1}^1 \\ &\quad + \int_{-1}^1 \frac{1}{1-x} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \cdot dx \\ &= \int_{-1}^1 \frac{1}{1-x} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \end{aligned}$$

because of the zeros of $\frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n$ at $x = \pm 1$. Integrating again

by parts

$$\begin{aligned} \int_{-1}^1 \frac{1}{1-x} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx &= \frac{1}{1-x} \cdot \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n \Big|_{-1}^1 - \int_{-1}^1 \frac{1}{(1-x)^2} \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n \cdot dx \\ &= - \int_{-1}^1 \frac{1}{(1-x)^2} \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n dx. \end{aligned}$$

Continuing in this way we eventually obtain

$$\int_{-1}^1 \frac{1}{1-x} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \cdot dx = (-1)^{n-1} (n-1)! \int_{-1}^1 \frac{1}{(1-x)^n} (x^2-1)^n dx.$$

Since $(x^2-1)^n = (x+1)^n \cdot (x-1)^n = (-1)^n \cdot (x+1)^n (1-x)^n$ we see that

$$\begin{aligned} \int_{-1}^1 \frac{1}{1-x} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx &= - (n-1)! \int_{-1}^1 (x+1)^n dx \\ &= - (n-1)! \frac{(x+1)^{n+1}}{n+1} \Big|_{-1}^1 \\ &= - (n-1)! \cdot \frac{2^{n+1}}{n+1} . \end{aligned}$$

Therefore

$$\begin{aligned} \int_{-1}^1 \log(1-x) P_n(x) dx &= \frac{1}{2^n n!} \cdot \frac{-(n-1)! 2^{n+1}}{n+1} \\ &= - \frac{2}{n(n+1)} . \end{aligned}$$

7. In polars the Laplace operator is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} .$$

In the unit disc

$$\phi(re^{i\theta}) = \operatorname{Re}[h(z)]$$

for some analytic function $h(z) = \sum_{k=0}^{\infty} a_k (re^{i\theta})^k$, with $z = r e^{i\theta}$.

Similarly consider $\phi(re^{i\theta}) = \operatorname{Re}[H(z)]$ with $H(z) = \sum_{k=0}^{\infty} A_k (re^{i\theta})^k$.

Now

$$\begin{aligned} \nabla^2 (r^2 \phi) &= \frac{\partial}{\partial r} \left(2r\phi + r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r} \left(2r\phi + r^2 \frac{\partial \phi}{\partial r} \right) + \frac{\partial^2 \phi}{\partial \theta^2} r \\ &= 2\phi + 4r \frac{\partial \phi}{\partial r} + r^2 \frac{\partial^2 \phi}{\partial r^2} + 2\phi + r \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2} r \end{aligned}$$

$$= r^2 \frac{\partial^2 \phi}{\partial r^2} + 5r \frac{\partial \phi}{\partial r} + 2 \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2}.$$

$$= \operatorname{Re} \left[\sum_{k=0}^{\infty} k(k-1) A_k (re^{i\theta})^k + 5 \sum_{k=0}^{\infty} k A_k (re^{i\theta})^k + 2 \sum_{k=0}^{\infty} A_k (re^{i\theta})^k - \sum_{k=0}^{\infty} k^2 A_k (re^{i\theta})^k \right]$$

$$= \operatorname{Re} \left[\sum_{k=0}^{\infty} (4k+2) A_k (re^{i\theta})^k \right].$$

Therefore if we define $A_k = \frac{a_k}{4k+2}$ ($k \geq 0$) we obtain a harmonic function ϕ such that

$$\nabla^2 (u - r^2 \phi) = \phi - \phi = 0$$

and so $u - r^2 \phi$ is real harmonic in the disc. There is a function $g(z)$ analytic in the disc so that

$$u - r^2 \phi = \operatorname{Re}[g(z)].$$

Also

$$r^2 \phi = \operatorname{Re}[r^2 H(z)] = \operatorname{Re}[\bar{z} \cdot z H(z)]$$

and so, defining $f(z) = zH(z)$, we find that

$$u = \operatorname{Re}[\bar{z}f(z) + g(z)].$$

8. Let $y(z) = \sqrt{x} u(x)$ and then

$$y'(x) = \frac{1}{2} x^{-1/2} u(x) + x^{1/2} u'(x),$$

$$y''(x) = -\frac{1}{4} x^{-3/2} u(x) + x^{-1/2} u'(x) + x^{1/2} u''(x).$$

Hence

$$-\frac{1}{4} x^{-3/2} u(x) + x^{-1/2} u'(x) + x^{1/2} u''(x) + \frac{1}{3} x^{3/2} u(x) = 0,$$

i.e.

$$x^{1/2} u''(x) + x^{-1/2} u'(x) + \left(\frac{1}{3} x^{3/2} - \frac{1}{4} x^{-3/2}\right) u(x) = 0.$$

Now let $\xi^2 = \frac{4x^3}{27}$ and set

$$u(x(\xi)) = v(\xi).$$

Then

$$\begin{aligned} \frac{du}{dx} &= \frac{d\xi}{dx} \cdot \frac{dv}{d\xi} \\ &= \frac{3}{\sqrt{27}} x^{1/2} \frac{dv}{d\xi} = \frac{3}{\sqrt{27}} \left(\frac{27}{4}\right)^{1/6} \xi^{1/3} \frac{dv}{d\xi}; \\ \frac{d^2 u}{dx^2} &= \frac{3}{2\sqrt{27}} x^{-1/2} \frac{dv}{d\xi} + \frac{9}{27} x \frac{d^2 v}{d\xi^2} \\ &= \frac{3}{2\sqrt{27}} \left(\frac{4}{27}\right)^{1/6} \xi^{-1/3} \frac{dv}{d\xi} + \frac{9}{27} \left(\frac{27}{4}\right)^{1/3} \xi^{2/3} \frac{d^2 v}{d\xi^2}. \end{aligned}$$

Therefore $v(\xi)$ satisfies the equation

$$\begin{aligned} \left(\frac{27}{4}\right)^{1/6} \xi^{1/3} \left\{ \frac{3}{2\sqrt{27}} \left(\frac{4}{27}\right)^{1/6} \xi^{-1/3} v' + \frac{9}{27} \left(\frac{27}{4}\right)^{1/3} \xi^{2/3} v'' \right\} \\ + \left(\frac{4}{27}\right)^{1/6} \xi^{-1/3} \cdot \frac{3}{\sqrt{27}} \left(\frac{27}{4}\right)^{1/6} \xi^{1/3} v' \\ + \left\{ \frac{1}{3} \left(\frac{27}{4}\right)^{1/2} \cdot \xi - \frac{1}{4} \cdot \left(\frac{4}{27}\right)^{1/2} \cdot \xi^{-1} \right\} v = 0 \end{aligned}$$

and so

$$\frac{9}{27} \cdot \left(\frac{27}{4}\right)^{1/2} \cdot \xi v'' + \frac{9}{2\sqrt{27}} v' + \frac{1}{3} \left(\frac{27}{4}\right)^{1/2} \left\{ \xi - \frac{3}{4} \cdot \frac{4}{27} \xi^{-1} \right\} v = 0,$$

i.e.

$$\xi^2 v'' + \xi v' + \left(\xi^2 - \left(\frac{1}{3}\right)^2\right) v = 0$$

This is Bessel's equation of order $\frac{1}{3}$ and has the general solution

$$v(\xi) = A J_{1/3}(\xi) + B J_{-1/3}(\xi).$$

Consequently

$$Y = \sqrt{x} \cdot v(\xi) = \sqrt{x} (A J_{1/3}(\xi) + B J_{-1/3}(\xi)).$$

9. Note that the integral is the convolution of y and $\frac{1}{t^\alpha}$ and so the

Laplace transform of the integral is

$$L[y] \cdot L[t^{-\alpha}] = Y(s) \cdot \int_0^\infty t^{-\alpha} e^{-st} dt \quad (s > 0)$$

$$= Y(s) \cdot \frac{1}{s^{1-\alpha}} \cdot \int_0^\infty u^{(1-\alpha)-1} e^{-u} du$$

$$= Y(s) \cdot \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}.$$

Since

$$\int_0^\infty t e^{-st} dt = \frac{1}{s^2} \int_0^\infty u e^{-u} du = \frac{1}{s^2}$$

it follows that

$$Y(s) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{s^{1+\alpha}}$$

For $\beta > -1$,

$$\int_0^\infty t^\beta e^{-st} dt = \frac{1}{s^{1+\beta}} \int_0^\infty u^\beta e^{-u} du$$

$$= \frac{\Gamma(1+\beta)}{s^{1+\beta}}$$

and so taking $\beta = \alpha$ we see that $Y(s)$ is the transform of

$$y(t) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{1}{\Gamma(1+\alpha)} \cdot t^\alpha.$$

10. Consider the method of steepest descent as described in JWD, p.453. In the notation used there

$$h(\zeta) = \frac{i}{2}(\zeta - \frac{1}{\zeta})$$

and

$$h'(\zeta) = \frac{i}{2}(1 + \frac{1}{\zeta^2})$$

so that the saddle points occur at $\zeta = \pm i$. Also

$$\begin{aligned} \text{Im}[h(\zeta)] &= \frac{1}{2} \text{Re}[\zeta - \frac{\bar{\zeta}}{|\zeta|^2}] \\ &= \frac{1}{2}(\zeta - \frac{\xi}{|\zeta|^2}) \quad (\zeta = \xi + i\eta) \\ &= 0 \end{aligned}$$

at $\zeta = \pm i$. Now

$$\zeta - \frac{\xi}{|\zeta|^2} = \xi \frac{|\zeta|^2 - 1}{|\zeta|^2}$$

and so one sees that the steepest descent curve at $\zeta = \pm i$ is the η -axis or the unit circle. One hopes to be able to take the unit circle since the integral allows such a choice for C , by Cauchy's theorem; and since $x \rightarrow \infty$ we hope to be able to pick whichever one of $\zeta = i, \zeta = -i$

is appropriate. Now if we put $\zeta = \pm i e^{i\phi}$, then

$$\begin{aligned} \frac{i}{2}(\zeta - \frac{1}{\zeta}) &= \pm \frac{i}{2}(i e^{i\phi} - \frac{1}{i e^{i\phi}}) \\ &= \pm \frac{e^{i\phi} + e^{-i\phi}}{2} \\ &= \pm \cos \phi. \end{aligned}$$

Therefore when $x \rightarrow \infty$ one sees that $\zeta = -i$ is the right choice.

Consider therefore C to be given by $\zeta = -i e^{i\phi} \quad (-\pi < \phi < \pi)$.

Then

$$\begin{aligned} I_n(x) &= \frac{e^{-in\frac{\pi}{2}}}{2\pi i} \int_{-\pi}^{\pi} e^{i(n+1)\frac{\pi}{2}} \cdot e^{-(n+1)i\phi} \cdot e^{xcos\phi} \cdot e^{i\phi} d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ni\phi} \cdot e^{xcos\phi} d\phi. \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos n\phi - i \sin n\phi) e^{xcos\phi} d\phi \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos n\phi e^{xcos\phi} d\phi \end{aligned}$$

Since $\sin n\phi \cdot e^{xcos\phi}$ is an odd function of ϕ . Hence

$$I_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos n\phi \cdot e^{xcos\phi} d\phi$$

since $\cos n\phi \cdot e^{xcos\phi}$ is an even function of ϕ . Put $\cos\phi = q - t$ and we obtain

$$I_n(x) = \frac{1}{\pi} \int_0^2 p(t) e^{x(1-t)} \cdot \frac{dt}{\sqrt{2t} \cdot \sqrt{1-\frac{t}{2}}}$$

where $p(t)$ is a polynomial in t and $p(0) = 1$. Hence, as $x \rightarrow \infty$,

$$\begin{aligned} I_n(x) &\sim \frac{1}{\sqrt{2\pi}} \cdot e^x \int_0^{\infty} t^{-1/2} e^{-xt} dt \\ &= \frac{1}{\sqrt{2\pi}} \cdot e^x \cdot \frac{1}{\sqrt{x}} \int_0^{\infty} u^{\frac{1}{2}-1} \cdot e^{-u} du \\ &= \frac{\Gamma(\frac{1}{2}) \cdot e^x}{\sqrt{2\pi} \cdot \sqrt{x}} \\ &= \frac{e^x}{\sqrt{2\pi x}}, \end{aligned}$$

since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (JWD, p.198).

